INVARIANT METRICS ON SOME SUBHOMOGENEOUS COMPLEX DOMAINS AND A RELATED COMBINATORIAL CONJECTURE

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Among many invariant metrics on bounded complex domains in dimension $d > 1$, the Bergman metric and the complete Kaehler-Einstein metric (Cheng-Yau metric) have been computed explicitly by G. Roos, Weiping Yin and collaborators for a class of non-homogeneous domains, built through "inflation" of bounded symmetric domains. The comparison of the two above invariant metrics leads to a combinatorial conjecture about the signs of coefficients in the decomposition of the Hua polynomial along raising factorials. Up to now, this conjecture is proved in special cases and has also been verified with help of computer algebra software in many significant cases.

Let Ω be a bounded irreducible symmetric domain. We denote by N the generic norm of Ω, by d its complex dimension, by γ the genus of Ω . Let $\mu > 0$ be a real number and

$$\widetilde{\Omega} = \widetilde{\Omega}(\mu) = \left\{(z,w) \in \Omega \times \mathbb{C} \mid |w|^2 < N(z,z)^{\mu}\right\}.$$

The Bergman kernel and the Kähler-Einstein metric are computed for the non-homogeneous domain $\widetilde{\Omega}$.

For $\mu_0 = \frac{\gamma}{d+1}$, the Kähler-Einstein metric of $\widetilde{\Omega}\,(\mu_0)$ is associated to the Kähler form

$$\partial\overline{\partial}g = -\partial\overline{\partial}\log\left(N(z,z)^{\mu_0} - |w|^2\right).$$

We call μ_0 the *critical exponent* for the bounded symmetric domain Ω.

For the critical exponent μ_0, it seems that the Bergman kernel of $\widetilde{\Omega}\,(\mu_0)$ also has special properties.

Let Ω be an irreducible bounded circled homogeneous domain, with numerical invariants a, b, rank r and genus $\gamma = 2 + a(r-1) + b$. The *Hua polynomial* of Ω is

$$\chi(s) = \prod_{j=1}^{r}\left(s + 1 + (j-1)\frac{a}{2}\right)_{1+b+(r-j)a}.$$

Let

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{d} c_{\mu,j}\frac{(k+1)_j}{j!},$$

where $(k+1)_j$ denotes the raising factorial $\Gamma(k+j)/\Gamma(k)$.

Conjecture 1 *For $\mu = \mu_0$, all coefficients $c_{\mu,j}$ in*

$$\chi(\mu s) = \sum_{j=0}^{d} c_{\mu,j}(s+1)_j.$$

are strictly positive, *except $c_{\mu,d-1} = 0$ and except for the type $I_{1,n}$ (where $c_{\mu,d} = 1$ and $c_{\mu,j} = 0$ for all $j < d$).*

The conjecture has been proved in special cases; it has also been checked in many cases with help of computer algebra software. Moreover, experiments with computer algebra seem to indicate that the conjecture holds even if the triple (a, b, r) is not related to a bounded symmetric domain.